BUCKLING AND POST-BUCKLING BEHAVIOUR OF THERMALLY LOADED SKIN PANELS

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Summary—The paper considers skin panels of multi-web wings under longitudinal, thermal and compressive strain.

Non-uniformity of the stress in the edge region appears to have a negligible effect on the buckling mode. This yields a straightforward method for establishing buckling loads if the solution for uniform stress is available.

There appears to be a negligible difference between the post-buckling behaviour in the thermal case and the behaviour at constant temperature for the practical range of ratios between thermal and buckling strain; the wave pattern is almost unaffected by thermal strain. Hence the relation between load increments and edge displacement increments after the onset of buckling is equal for thermal and isothermal conditions.

For panels subjected to thermal and compressive strain, data are given on load, wave depth, wavelength and stresses as functions of edge strain. Data on the isothermal tension field are also applicable to thermal conditions.

NOTATION

Apart from some notations of local importance, which are defined in the text, this report contains the following notations of more general interest.

a=panel length
b=panel width
f=wave depth
k=buckling coefficient for thermal stress distribution
k_0=buckling coefficient for uniform stress
t=plate thickness
$p=\pi x/L$
$q=\pi y/\beta b$
u, v=displacements of the middle surface of the plate parallel to the x- and y-axes
w=deflection of the plate
x, y=longitudinal and lateral co-ordinates
A_i=coefficients of Eq. (4.8) depending on the wave pattern (see 4.7)
$B=Et^3[12(1-\nu^2)]^{-1}$, bending stiffness of the plate
$D=(b/L)^2$, wavelength parameter

$F = \frac{\pi^2}{4}(f/b)^2$, wave depth parameter

L = half wave length
P = longitudinal compressive load
$\overline{P}$ = compressive load for constant temperature
T = local temperature, function of y
$W(y)$ = cross shape of the wave pattern, defined by Eq. (4.5)
X_i, Y_i, Z = functions of $\bar{\epsilon}/\epsilon_0$, given in Table IV and Figs. 6 and 7
α = coefficient of thermal expansion
β = ratio between the width of the double curved edge region and b
ϵ = compressive strain of the edge
$\epsilon_0 = \pi^2(t/b)^2[3(1-\nu^2)]^{-1}$, buckling strain for constant temperature
$\epsilon_T(\alpha T, \beta)$ = function of temperature and wave shape, defined by Eq. (4.9)
$\bar{\epsilon}$ = apparent strain, defined by Eq. (4.10), strain for constant temperature
σ_x, σ_y = normal stresses
σ_a = equivalent stress, defined by (4.21)
τ = shear stress
$(\alpha T)_{av}$ = average thermal strain in the panel.
b as suffix denotes the onset of buckling.

1. INTRODUCTION

The structure of supersonic aircraft is critically loaded as far as thermal stresses are considered during the so-called transient conditions, when the aircraft is accelerated. Kinetic heating of the skin while the interior structure is still at a lower temperature level causes compressive stresses in the skin and tensile stresses in the internal structure. The temperature of the skin is not constant or nearly constant. Those places where the skin is in contact with the interior structure act as heat sinks and have a considerably lower temperature than the skin at some distance from the joint.

The problem investigated in this paper refers to the multi-web wing structure, which is very common for supersonic aircraft. The type of temperature distribution occurring in the skin of such wings is illustrated by Fig. 1. Since the longitudinal strain of the skin panel is constant throughout the plate due to the compatibility of strains and because the resultant load on the wing section should vanish (leaving the aerodynamic loads for the time being out of consideration) compressive stresses occur in the centre part of the panels and tensile stresses in the edge regions of the panel and in the interior structure. The magnitude of these stresses depends on the longitudinal stiffness of the longitudinal webs of the wing. Since this

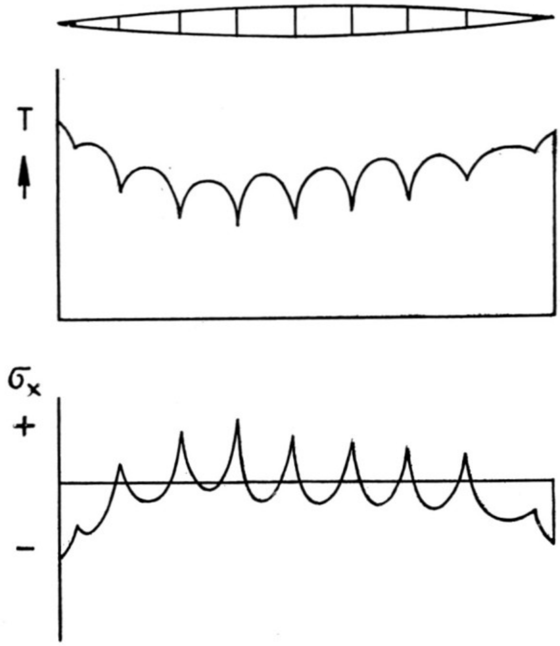

FIG. 1. Multi-web wing. Temperature distribution and thermal stresses in transient condition.

is a measure to restrict the thermal compressive stresses of the skin it is advantageous to design the webs in such a manner that their longitudinal stiffness is small. Corrugated webs are very useful in this respect. Though the magnitude of the thermal stresses is reduced by such measures the stress differences within an individual panel are not affected. The longitudinal stresses in a panel are far from homogeneous (see Fig. 1) and this fact gives rise to a type of buckling problem, which did not exist for subsonic aircraft.

The buckling problem to be investigated is not the problem of the plate merely loaded by thermal stresses, the average of which over the plate width is near zero. To the thermal stresses are added the stresses imposed by the aerodynamic load of the aircraft. These latter stresses are practically constant over the width of the panel: they consist mainly of compressive (or tensile) stresses in the longitudinal direction.

Therefore the buckling problem to be considered is that of a panel, the length of which is a great multiple of its width, supported along its edges and loaded longitudinally by compressive stresses which are maximal in the centre of the panel and drop off to a lower value at the edges. This stress distribution is characterized by a steep gradient at the edge and an almost constant part in the centre region of the panel. An example illustrating the temperature and the corresponding stress distribution has been given by Hoff[1]. Since the adjacent panels will have about the same size and will be loaded in about the same manner the interaction of adjoining panels can be supposed to be negligible. From this point of view the edges of the panel can be considered to be simply supported. The possibility exists that the webs will present some restraint against panel buckling. In this investigation, however, such restraint is being neglected.

The stability of plates under non-uniform stress of this type has been investigated by Hoff[1]. He gives an evaluation for the stress distribution $\sigma_0+\sigma_1\cos 2\pi y/b$. In ref. 2 the case $\sigma_0+\sigma_1\sin\pi y/b$ is evaluated and some other distributions characterized by a large stress gradient at the edges were investigated by means of a so-called "correction method" which will be recapitulated in Section 2 of this paper. The basic idea of this method suggests the approach to the problem of the post-buckling behaviour of thermally strained plates, which is the main subject of this paper.

Surface waviness of the airfoil in normal flight is without doubt prohibitive since it increases its drag. However, increased drag occurring only during a small part of the total flight time might well be acceptable as a compromise between considerations on aerodynamic cleanness and structural weight. So for instance might buckling be accepted under the combined action of normal flight loads and thermal stresses during the acceleration period, and even more if manoeuvring loads are added.

The requirement that buckling should not occur in certain conditions of flight creates the need for methods to establish buckling loads. The allowability of buckling in other conditions of flight creates the need for knowledge on post-buckling behaviour. Information is required on the following subjects:

1. The relation between the deformation of the panel and its load. This information is needed for the stress analysis of the aircraft.
2. The stiffness of the panel in its post-buckling state with respect to small variations of the deformation.
   This information is needed for investigations on the possibility of flutter.
3. The maximal stresses in the buckled panel.
   After having sustained large thermal and/or aerodynamic load, which causes buckling, the skin panel should return in its undeflected position. Therefore the maximal stresses occurring with buckling should not cause permanent deformation and their magnitude should be known.
4. Quantitative data on the shape of the buckled panel.
   This information will be required by the aerodynamicist, since surface waviness should be kept below certain limits in order to avoid turbulence of such a violence that it would be detrimental to the structure.

This paper supplies information on these four items for the simply supported panel under thermal load and longitudinal compression, on items 1, 2 and 4 for the rigidly restrained panel under thermal load and longitudinal compression. Finally it deals with simply supported panels under thermal load and the combined action of longitudinal and lateral compression and shear. In this respect available information enables us to answer questions with respect to items 1, 2 and 3.

2. BUCKLING OF SKIN PANELS

The differential equation for buckling of plates of uniform thickness is[3]

$$D(w) \equiv B\nabla^4 w - t\left(\sigma_x \frac{\partial^2 w}{\partial x^2} + 2\tau \frac{\partial^2 w}{\partial x \partial y} + \sigma_y \frac{\partial^2 w}{\partial y^2}\right) = 0, \tag{2.1}$$

where

$$\nabla^4(\ ) \equiv \frac{\partial^4(\ )}{\partial x^4} + 2\frac{\partial^4(\ )}{\partial x^2 \partial y^2} + \frac{\partial^4(\ )}{\partial y^4}$$

This equation has, in the case of the rectangular plate supported along its edges and uniformly loaded in longitudinal compression, the very simple solution

$$w = f \sin \pi y/b \sin \pi x/L. \tag{2.2}$$

Under thermal conditions the longitudinal stress σ_x varies across the plate width; one of the coefficients of the differential Eq. (2.1) is no longer constant and the problem is mathematically more complex. Then the exact solution can be obtained by infinite series[1,2]. To this end σ_x is expanded in Fourier-series, in ref. 1 as

$$-\sigma_x t = S_0(1 + \Sigma \mu_p \cos p\pi y/b) \qquad p = 2,4,6, \tag{2.3a}$$

in ref. 2 as

$$-\sigma_x t = S_0(1 + \Sigma \mu_p \sin p\pi y/b) \qquad p = 1,3,5, \tag{2.3b}$$

and w is expanded in Fourier-series as

$$w = (\Sigma f_n \sin n\pi y/b) \sin \pi x/L \qquad n = 1,3,5 \tag{2.4}$$

This exact solution becomes an approximate solution when the number of terms considered in (2.4) is restricted. Then the best approximation for the coefficients f_n of the finite series are obtained by the application of the Ritz–Galerkin method[2].

When the approximation for the buckling mode is denoted as

$$\overline{w} = \Sigma f_n w_n(x,y) \tag{2.5}$$

the Ritz–Galerkin-equations replacing Eq. (2.1) are

$$\int_0^a \int_0^b D(\overline{w}) w_j \mathrm{d}x \mathrm{d}y = 0, \tag{2.6}$$

where w_j is any of the functions composing the finite series (2.5).

Equations (2.6) are linear in the unknown coefficients f_n; they define a characteristic value problem. Its solution is readily obtained by matrix iteration, which yields with the vector f the characteristic value $1/S_0$. Since usually one is interested only in the first buckling load, which corresponds to the largest characteristic value, the evaluation of the Eqs. (2.6) need not be continued after having obtained the first characteristic value.

Reference 2 gives as an example the solution for the load

$$\sigma_x t = S_0(1 - 15/8 \sin \pi y/b - 5/24 \sin 3\pi y/b).$$

The stress σ_x in this case is distinctly non-uniform; the edges are loaded in tension, whereas the centre is loaded in compression; the ratio of both stresses is 1·5. The series (2.5) include only three terms. Starting the iteration with $f_1=1$, $f_3=0$, $f_5=0$ one obtains after one iteration step $f_1=1$, $f_3=-0{\cdot}0131$, $f_5=-0{\cdot}0009$ and after the second one $f_1=1$, $f_3=-0{\cdot}0133$, $f_5=-0{\cdot}0009$.

The remarkable feature emanating from this example is that although the load is far from uniform the buckling mode departs only slightly from that for uniform load. Therefore the buckling mode for uniform load is a reasonable approximation for the mode pertaining to non-uniform loads of the type prevailing in kinetic heated skin panels. This conclusion is reflected in the buckling stresses. The buckling load obtained when only the first term of the series (2.4) is maintained is only 0·6% in error, against the buckling load obtained by matrix iteration on the basis of three terms of the series (2.4).

These conclusions have a simple physical explanation. The energy required for distorting the plate into its buckling mode is supplied by the external load at the edges. The external work is[3]:

$$A_e = -\tfrac{1}{2}t \int_0^a \int_0^b \left\{ \sigma_x \left(\frac{\partial w}{\partial x}\right)^2 + 2\tau \frac{\partial w}{\partial x} \cdot \frac{\partial w}{\partial y} + \sigma_y \left(\frac{\partial y}{\partial y}\right)^2 \right\} \mathrm{d}x\mathrm{d}y, \qquad (2.7)$$

where for sake of generality τ and σ_y are included, though they were zero in the case considered so far.

At the edges the panel is supported, hence the deflection w vanishes at the edge and is very small near the edge in those parts of the panel where the stress σ_x departs sensibly from uniformity. Since (2.7) is quadratic in w and $\partial w/\partial x$ is small near the edge the contribution of σ_x to the external work is small as far as the edge regions are concerned. Therefore the peculiarity of the distribution of σ_x will affect the external work and consequently the strain energy of the plate only slightly. The energy is only slightly affected by the non-uniformity of the stresses and consequently the buckling mode must be close to that for uniform stresses.

This conclusion suggests a straightforward approximation for the type of buckling problems concerned.

Let us suppose that the buckling problem for uniform stress distribution $\sigma_x = -s_0$, $\sigma_y = -C_1 s_0$, $\tau = -C_2 s_0$ has been solved, that its mode is $w(x,y)$ and the buckling load is $s_0 = k_0 \pi^2 B/b^2 t$, where k_0 is the buckling coefficient.

Then the external work is, according to Eq. (2.7),

$$A_{e0} = \tfrac{1}{2} \int_0^a \int_0^b \left\{ \left(\frac{\partial w}{\partial x}\right)^2 + 2C_2 \frac{\partial w}{\partial x} \cdot \frac{\partial w}{\partial y} + C_1 \left(\frac{\partial w}{\partial y}\right)^2 \right\} \mathrm{d}x\mathrm{d}y \cdot k_0 \pi^2 B/b^2 \quad (2.8)$$

The non-uniform stress distribution which yields buckling is

$$\sigma_x = -s + \bar{s},\ \sigma_y = -C_1 s,\ \tau = -C_2 s,$$

where s is constant throughout the plate and is equal to $-\sigma_x$ in the centre of the panel, and $\bar{s}$ represents the non-uniformity of σ_x in the vicinity of the edges (Fig. 2). The buckling coefficient is defined by

$$s = k\pi^2 B/b^2 t.$$

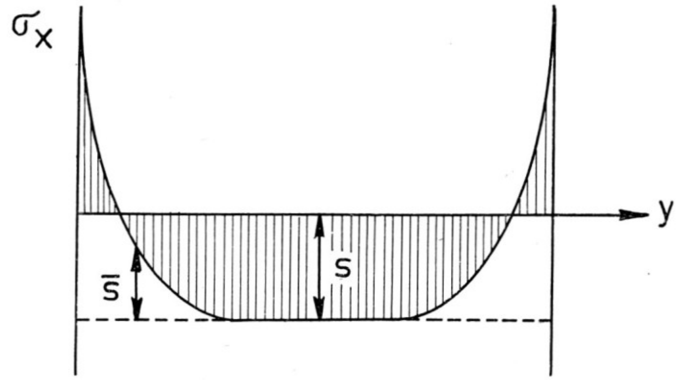

FIG. 2. Distribution of thermal stresses over panel width.

In this thermal case the external work is according to Eq. (2.7)

$$A_e = \tfrac{1}{2} \int_0^a \int_0^b \left[\left(1 - \frac{\bar{s}}{s}\right) \left(\frac{\partial w}{\partial x}\right)^2 + 2C_2 \frac{\partial w}{\partial x} \cdot \frac{\partial w}{\partial y} + C_1 \left(\frac{\partial w}{\partial y}\right)^2 \right] \mathrm{d}x\mathrm{d}y \times k\pi^2 B/b^2 \quad (2.9)$$

On the assumption that the modes may be considered to be equal for uniform stresses and non-uniform stresses it follows that the elastic energies in both cases are equal and consequently that the external work in these cases is equal : $A_e = A_{e0}$.

Equating (2.8) and (2.9) yields

$$\frac{k - k_0}{k} \cdot \int_0^a \int_0^b \left[\left(\frac{\partial w}{\partial x}\right)^2 + 2C_2 \frac{\partial w}{\partial x} \cdot \frac{\partial w}{\partial y} + C_1 \left(\frac{\partial w}{\partial y}\right)^2 \right] \mathrm{d}x\mathrm{d}y = \int_0^a \int_0^b \bar{s}/s \left(\frac{\partial w}{\partial x}\right)^2 \mathrm{d}x\mathrm{d}y \quad (2.10)$$

Since $k_0, w(x,y)$ and $\frac{1}{s}\bar{s}(y)$ are known, equation (2.10) yields the buckling coefficient k.

Equation (2.10) defines the correction which must be applied to k_0 for uniform load to find k for non-uniform load. Therefore the method might be called the "correction method".

This method has been applied in ref. 2 to some distributions of thermal stress with steep gradients at the edges.

3. POST-BUCKLING BEHAVIOUR OF SKIN PANELS

The differential equations governing the post-buckling behaviour of plates are non-linear in the deflections. This augments the difficulty, already present due to anisothermic conditions, of obtaining exact solutions and one is compelled to use approximative methods of solution.

The method here applied is the Raleigh–Ritz method. Thus approximations for the three displacement components are assumed, which contain a limited number of parameters. These parameters are chosen in such a way that strain energy is minimal as a function of the parameters. Since the strain energy established in this way is an approximation of the actual minimum of strain energy its amount is somewhat too large. Consequently, the external load corresponding to this energy is more or less an over-estimation of the actual load.

The method has been applied by Koiter in his work on the iso-thermal case of plates loaded in longitudinal compression, the "effective-width" problem(4), and on plates loaded simultaneously by longitudinal and lateral compression and shear(5). Experimental evidence exists to show that the over-estimation inherent in the approximation is negligible.

In the investigation which follows, the line of thought of ref. 4 is applied to the longitudinally-compressed simply-supported plate with non-homogeneous and with respect to the centreline symmetrical temperature distribution.

The adaptation of the general equations to the case of non-homogeneous temperature precedes.

The relations between the membrane strains, the membrane stresses and temperature are

$$\begin{aligned} E\epsilon_x &= \sigma_x - \nu\sigma_y + E\alpha T \\ E\epsilon_y &= \sigma_y - \nu\sigma_x + E\alpha T \\ E\gamma &= 2(1+\nu)\tau \end{aligned} \tag{3.1}$$

The relation between the curvatures and the moments are independent of temperature.

$$\begin{aligned} B(1-\nu^2)\frac{\partial^2 w}{\partial x^2} &= M_x - \nu M_y \\ B(1-\nu^2)\frac{\partial^2 w}{\partial y^2} &= M_y - \nu M_x \\ B(1-\nu^2)\frac{\partial^2 w}{\partial x \partial y} &= (1+\nu)M_{xy} \end{aligned} \tag{3.2}$$

Since the Raleigh–Ritz method is an energy method we have to establish the strain energy. The strain energy per unit of area of the plate and over its full thickness, is

$$A = \tfrac{1}{2} \left\{ t\,(\sigma_x \epsilon_x + \sigma_y \epsilon_y + \tau\gamma) + M_x \frac{\partial^2 w}{\partial x^2} + M_y \frac{\partial^2 w}{\partial y^2} + 2M_{xy} \frac{\partial^2 w}{\partial x \partial y} \right\}. \quad (3.3)$$

Substitution of (3.1) and (3.2) into (3.3) yields

$$A = \tfrac{1}{2} \frac{Et}{1-\nu^2} \Big((\epsilon_x + \epsilon_y)^2 - 2(1-\nu)\,(\epsilon_x \epsilon_y - \frac{1}{4}\gamma^2) - $$

$$- 2\,(1+\nu)\,\alpha T\,(\epsilon_x + \epsilon_y) + 2\,(1+\nu)\,(\alpha T)^2 +$$

$$+ \frac{1}{12} t^2 \left[\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 - 2\,(1-\nu) \left\{ \frac{\partial^2 w}{\partial x^2} \cdot \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right\} \right] \Big) \quad (3.4)$$

Equation (3.4) expresses the strain energy as a function of the displacement components u, v, w, by means of the relations between displacements and strains valid for finite deflections w

$$\epsilon_x = \frac{\partial u}{\partial x} + \tfrac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2$$

$$\epsilon_y = \frac{\partial v}{\partial y} + \tfrac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 \quad (3.5)$$

$$\gamma = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \cdot \frac{\partial w}{\partial y}.$$

4. POST-BUCKLING BEHAVIOUR UNDER LONGITUDINAL COMPRESSION

4.1 *The Assumptions on the Displacements with Simply Supported Edges*

We have already found that the buckling mode in the isothermal case is a very good approximation for the buckling mode under thermal conditions. This applies to the mode at the onset of buckling. If the longitudinal compression increases the deflections become finite. Whereas infinitesimal deflections correspond to the buckling mode, finite deflections may be incongruent to the buckling mode, though it can be expected that the deviation from exact congruency will be slight in the first post-buckling stage. Therefore, though it is an approximation, the function

$$w = f \sin \pi y/b \sin \pi x/L$$

is appropriate to describe the deflections in the isothermal case during the first post-buckling stage. Then, since this function represents a close approximation of the buckling mode in the thermal case, it will do as well as an approximation for the finite deflections in this first stage.

In the more advanced post-buckling stage, when the wave depth increases the wave pattern tends to become a developable surface[6]. This is a consequence of the fact that the membrane strain ("extensional") energy tends to become very large at large wave amplitudes. Then the "wash-board" pattern of the developable surface opens the possibility to obtain large deflections and nevertheless small membrane stresses.

In the edge regions the developable surface cannot be maintained since the deflection must vanish at the edges. Therefore the edge regions cannot be developable.

In accordance with this concept Cox[7] introduced a wave pattern consisting of a developable part over a width $(1-\beta)b$ in the centre of the plate and a double curved part over a width $\frac{1}{2}\beta b$ along the edges:

$$\tfrac{1}{2}\beta b < y < (1-\tfrac{1}{2}\beta)\, b \; : w = f \sin \pi\, x/L \tag{4.1a}$$

$$0 < y < \tfrac{1}{2}\beta b \; : w = f \sin \pi\, y/\beta b \sin \pi\, x/L \tag{4.1b}$$

This assumption for the deflection has been adopted by Koiter[4]. Together with consistent assumptions on u and v it yields a good approximation for the isothermal post-buckling behaviour. For the reasons already discussed it may then be applied with confidence in the thermal case.

The assumptions for u and v used in ref. 4 are

$$\tfrac{1}{2}\beta b < y < (1-\tfrac{1}{2}\beta)\, b \; : u = -\epsilon x - \frac{\pi}{8}\frac{f^2}{L}\sin 2\pi\frac{x}{L}\,;\; v = v_1(y) \tag{4.2a}$$

$$\left.\begin{aligned} 0 < y < \tfrac{1}{2}\beta b \qquad & : u = -\epsilon x - \frac{\pi}{8}\frac{f^2}{L}\sin^2 \pi\, y/\beta b \sin 2\pi\frac{x}{L} \\ & v = v_0(y) - \frac{\pi}{8}\frac{f^2}{\beta b}\sin 2\pi\, y/\beta b \sin^2 \pi\frac{x}{L}\,. \end{aligned}\right\} \tag{4.2b}$$

These functions satisfy the principle that the membrane stresses have to be small. They are such that $\gamma=0$ throughout, ϵ_x is a function of y only. The functions $v_0(y)$ and $v_1(y)$ follow from the condition that the average of σ_y over x vanishes.

$$\int_0^L \sigma_y \mathrm{d}x = 0 \tag{4.3}$$

Adopting the same characteristics for the membrane strains in the thermal case we obtain the same functions u and v, and the strains are expressed by the same formulae as those for the isothermal case.

The displacements, and consequently the strains, are functions of x and y and of the quantities f, L, β and ϵ.

For $y = 0$ (4.1b) (4.2b) yield

$$\frac{\partial w}{\partial x} = 0, \quad \frac{\partial u}{\partial x} = -\epsilon,$$

whereafter (3.5) yields

$$(\epsilon_x)_{y=0} = -\epsilon.$$

Therefore ϵ is the compressive strain at the edge. The problem to be solved now is to establish for a given ϵ the corresponding wave pattern parameters f, L and β, the wave depth, the half wavelength and the width of the double curved edge region. These three parameters are obtained from the condition that the strain energy is minimal.

4.2 *Minimal Strain Energy*

The formula for the strain energy per unit of area (3.4) departs from the expression valid in the isothermal case only by the addition of terms containing αT. If we denote the strain energy in the isothermal case ($T = 0$) by A_0 the strain energy in the thermal case is

$$A = A_0 + \frac{Et}{1-\nu}\left\{-\alpha T(\epsilon_x + \epsilon_y) + (\alpha T)^2\right\}$$

The total strain energy per half wavelength L is, since αT is a function of y only and ϵ_y can be eliminated by using Eq. (4.3)

$$S = \int_0^L\int_0^b A\,\mathrm{d}x\mathrm{d}y = S_0 - Et\int_0^b \mathrm{d}y\left(\alpha T\int_0^L \epsilon_x \mathrm{d}x\right) - EtL\frac{\nu}{1-\nu}\int_0^b(\alpha T)^2\mathrm{d}y$$

Introducing the average thermal strain

$$(\alpha T)_{\mathrm{av}} = \frac{1}{b}\int_0^b \alpha T\mathrm{d}y \tag{4.4}$$

and eliminating ϵ_x by means of (3.5) we obtain since $u_L - u_0 = -\epsilon L$

$$S = S_0 + EtbL\left[\epsilon(\alpha T)_{\mathrm{av}} - \frac{1}{2bL}\int_0^b \mathrm{d}y\left\{\alpha T\int_0^L\left(\frac{\partial w}{\partial x}\right)^2\mathrm{d}x\right\} + \right.$$
$$\left. + \frac{\nu}{1-\nu}\int_0^b(\alpha T)^2\mathrm{d}\,y/b\right]$$

If w is given by

$$w = fW(y)\sin\pi\, x/L, \tag{4.5}$$

which complies with (4.1) but applies as well to other assumptions on the wave pattern or to other edge conditions, and with the non-dimensional parameters F and D used in ref. 4, which replace f and L, the strain energy takes the form

$$S = S_0 + EtbL\left\{\epsilon(\alpha T)_{\mathrm{av}} - FD\int_0^b \alpha T W^2(y)\mathrm{d}\,y/b - \frac{\nu}{1-\nu}\int_0^b(\alpha T)^2\mathrm{d}\,y/b\right\}. \tag{4.6}$$

According to ref. 4 (Eq. 31.2) the strain energy for simply supported and restrained edges takes the form

$$S_0 = \tfrac{1}{2}EtbL \left\{\epsilon^2 - (2 - A_1\beta)\,\epsilon FD + F^2\left[(1 - A_2\beta)\,D^2 + A_2'\frac{D}{\beta} + \frac{A_3}{\beta^3}\right] + \right.$$

$$\left. + \,\epsilon_0 F\left[(\tfrac{1}{2} - A_4\beta)\,D^2 + A_5\frac{D}{\beta} + \frac{A_6}{\beta^3}\right]\right\},$$

where ϵ_0 is the buckling strain with simply supported edges and the coefficients A_i depend on the functions for u, v and w. For u, v, w defined by (4.1) (4.2) these coefficients are

$$A_1 = 1,\ A_2 = 5/8,\ A_2' = 0,\ A_3 = \frac{1}{8\,(1-\nu^2)} = 0{\cdot}1374,$$

$$A_4 = \frac{1}{4},\ A_5 = \tfrac{1}{2},\ A_6 = \frac{1}{4}. \tag{4.7}$$

Since A_1 has particular importance for this investigation its general formula is given

$$2 - A_1\beta = 2\int_0^b W^2(y)\,\mathrm{d}y/b.$$

Then the average strain energy per unit of length of the plate is the following function of the compressive overall strain ϵ the thermal strain αT and the parameters F, D, β.

$$\frac{S}{L} = \tfrac{1}{2}Etb\left[\epsilon^2 + 2\epsilon\,(\alpha T)_{\text{av}} - FD\,(2 - A_1\beta)\left\{\epsilon + \epsilon_T\,(\alpha T,\beta)\right\} + \right.$$

$$+ F^2\left\{(1 - A_2\beta)\,D^2 + A_2'\frac{D}{\beta} + \frac{A_3}{\beta^3}\right\} +$$

$$\left. + \,\epsilon_0 F\left\{(\tfrac{1}{2} - A_4\beta)\,D^2 + A_5\frac{D}{\beta} + \frac{A_6}{\beta^3}\right\} - \frac{2\nu}{1-\nu}\int_0^b (\alpha T)^2\mathrm{d}\,y/b\right] \tag{4.8}$$

where

$$\epsilon_T\,(\alpha T, \beta) = \frac{2}{2 - A_1\beta}\int_0^b \alpha T W^2(y)\,\mathrm{d}\,y/b =$$

$$= \int_0^b \alpha T W^2(y)\,\mathrm{d}\,y/b : \int_0^b W^2(y)\,\mathrm{d}y/b. \tag{4.9}$$

Since at a given edge strain ϵ the parameters F, D, β have to be such that strain energy is minimal the derivatives of S/L with respect to F, D and β should vanish.

If we replace in the equations the edge strain ϵ by the "apparent strain" $\bar{\epsilon}$, defined by

$$\bar{\epsilon} = \epsilon + \epsilon_T(\alpha T, \beta) \tag{4.10}$$

the equations from which F, D and β can be solved are

$$(\tfrac{1}{2}Etb)^{-1}\frac{\partial S/L}{\partial F} = -D\bar{\epsilon}(2 - A_1\beta) + 2F\left\{(1 - A_2\beta)D^2 + A_2'\frac{D}{\beta} + \frac{A_3}{\beta^3}\right\} + $$
$$+\epsilon_0\left\{(\tfrac{1}{2} - A_4\beta)D^2 + A_5\frac{D}{\beta} + \frac{A_6}{\beta^3}\right\} = 0 \tag{4.11a}$$

$$(\tfrac{1}{2}EtbF)^{-1}\frac{\partial S/L}{\partial D} = -\bar{\epsilon}(2 - A_1\beta) + F\left\{2(1 - A_2\beta)D + \frac{A_2'}{\beta}\right\} + $$
$$+\epsilon_0\left\{(1 - 2A_4\beta)D + \frac{A_5}{\beta}\right\} = 0 \tag{4.11b}$$

$$(\tfrac{1}{2}EtbF)^{-1}\frac{\partial S/L}{\partial \beta} = D\left\{\bar{\epsilon}A_1 - (2 - A_1\beta)\frac{\partial\epsilon_T}{\partial\beta}\right\} + F\left\{A_2D^2 + A_2'\frac{D}{\beta^2} + \right.$$
$$\left. + 3\frac{A_3}{\beta^4}\right\} - \epsilon_0\left\{A_4D^2 + A_5\frac{D}{\beta} + \frac{3A_6}{\beta^4}\right\} = 0. \tag{4.11c}$$

Let us suppose that we have solved for a given $\bar{\epsilon}$ the parameters F, D, β from Eqs. (4.11), then we can establish the longitudinal compressive load P.

The strain energy is produced by P over the displacement ϵa. Therefore

$$P = \frac{1}{a}\frac{\mathrm{d}}{\mathrm{d}\epsilon}\left(\frac{S}{L}\cdot a\right) = \frac{\partial S/L}{\partial\epsilon} + \frac{\partial S/L}{\partial F}\cdot\frac{\mathrm{d}F}{\mathrm{d}\epsilon} + \frac{\partial S/L}{\partial D}\cdot\frac{\mathrm{d}D}{\mathrm{d}\epsilon} + \frac{\partial S/L}{\partial\beta}\cdot\frac{\mathrm{d}\beta}{\mathrm{d}\epsilon}.$$

According to Eqs. (4.11) this reduces to

$$P = \frac{\partial S/L}{\partial\epsilon} = Etb\left\{\epsilon - FD(1 - \tfrac{1}{2}A_1\beta) + (\alpha T)_{\mathrm{av}}\right\}. \tag{4.12}$$

4.3 *Evaluation of the Solution*

Equations (4.11a and b) are identical to those for the isothermal case at the edge strain $\bar{\epsilon}$.

In the isothermal case β proves to be constant and equal to unity in the range of strain ratios $1 < \bar{\epsilon}/\epsilon_0 < 4$. In this range β is not a parameter of the wave pattern, hence (4.11c) cancels and F and D follow from (4.11a and b). Consequently F and D are equal in the thermal and the isothermal case for equal apparent strain $\bar{\epsilon}$. This yields a very simple relation between the load P in the thermal case and the load $\bar{P}$ in the isothermal case.

Since

$$\bar{P} = Etb\,[\bar{\epsilon} - FD(1 - \tfrac{1}{2}A_1\beta)]$$

we find

$$\bar{P} = P + Etb\,[\epsilon_T(\alpha T) - (\alpha T)_{\mathrm{av}}] \tag{4.13a}$$

further

$$\bar{\epsilon} = \epsilon + \epsilon_T(\alpha T). \tag{4.13b}$$

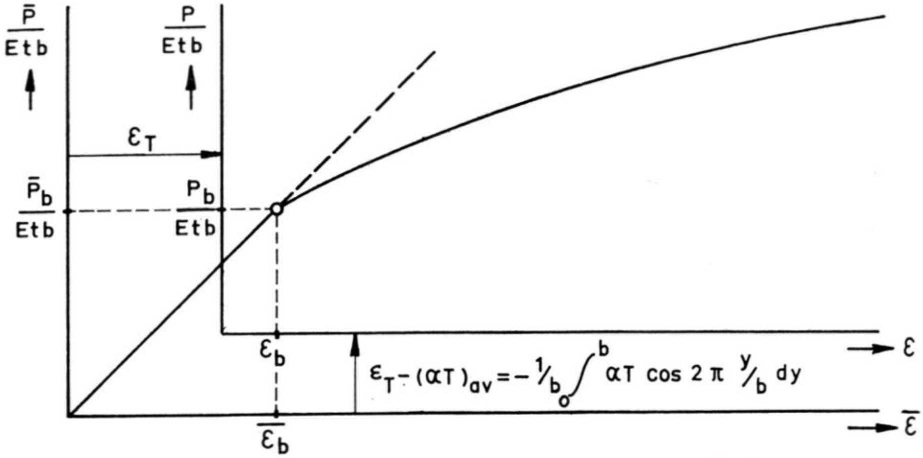

FIG. 3. Load P vs. edge strain ϵ for thermal conditions. $\overline{P}$, $\bar{\epsilon}$ applies to constant temperature.

This result has a simple graphical representation (Fig. 3). If the $\overline{P}$-$\bar{\epsilon}$-curve for the isothermal case is known, the P-ϵ curve for the thermal case is obtained by shifting the $\overline{P}$-axis in the direction of the $\bar{\epsilon}$-axis over the distance $\epsilon_T(\alpha T)$ and moving the $\bar{\epsilon}$-axis in the direction of the $\overline{P}$-axis over the distance

$$Etb\,\{\epsilon_T\,(\alpha T) - (\alpha T)_{\,\mathrm{av}}\}.$$

The graphical representation has among its characteristics that the representation of the buckling loads $\overline{P}_b$ and P_b of the isothermal and thermal case coincide. This yields another formulation of the correspondence of post-buckling behaviour in both cases.

If $\overline{P}_b$ and P_b are the buckling loads in the isothermal and the thermal case respectively, and $\bar{\epsilon}_b$, ϵ_b are the corresponding edge strains, the load increments $\overline{P}-\overline{P}_b$ and $P-P_b$ are equal for equal strain increments $\bar{\epsilon}-\bar{\epsilon}_b$ and $\epsilon-\epsilon_b$, and the wave patterns are equal.

Thus far this conclusion holds only in the primary post-buckling range where $\beta\equiv1$. In the more advanced stage the differences of F, D, β have to be taken into account.

Equations (4.11) and (4.12) where A_i is given by (4.7) can be written as

$$\frac{\bar{\epsilon}}{\epsilon_0} = 2\,(1-\nu^2)\left(1-\frac{5}{8}\beta\right) D\left(D^2\beta^3 - \frac{1}{2-\beta}\right) + \tfrac{1}{2}D + \frac{1}{2\beta\,(2-\beta)} \qquad (4.14a)$$

$$\frac{F}{\bar{\epsilon}} = \frac{\epsilon_0}{\bar{\epsilon}}\,(1-\nu^2)\,\{(2-\beta)\,D^2\beta^3 - 1\}. \qquad (4.14b)$$

$$-\frac{6-5\beta}{8}(1-\nu^2)\,\beta^3D^4 + \frac{6-5\beta}{8}\left(\frac{1}{\beta}+\frac{1-\nu^2}{2-\beta}\right)D^2 +$$

$$+\left\{\frac{1-\beta}{(2-\beta)\,\beta^2} + (2-\beta)\frac{\mathrm{d}}{\mathrm{d}\beta}(\epsilon_T/\epsilon_0)\right\}D + \frac{3}{8\beta^4} = 0 \qquad (4.14c)$$

$$\frac{P}{Etb\epsilon_0}=\frac{\bar{\epsilon}}{\epsilon_0}\left\{1-\frac{F}{\bar{\epsilon}}D\left(1-\tfrac{1}{2}\frac{\beta}{\pi}\right)\right\}+\left(\frac{\alpha T}{\epsilon_0}\right)_{\text{av}}-\frac{\epsilon_T}{\epsilon_0}, \qquad (4.15)$$

where

$$(2-\beta)\frac{\mathrm{d}}{\mathrm{d}\beta}(\epsilon_T/\epsilon_0)=\frac{2}{2-\beta}\left(\frac{\alpha T}{\epsilon_0}\right)_{\text{av}}-\frac{1}{\pi}\int_0^{\pi}\frac{\alpha T}{\epsilon_0}\eta\sin\eta\,\mathrm{d}\eta+$$

$$+\frac{\beta}{2-\beta}\frac{1}{\pi}\int_0^{\pi}\frac{\alpha T}{\epsilon_0}(1+\cos\eta)\,\mathrm{d}\eta,$$

$$\eta=2\pi\, y/\beta b.$$

These equations are evaluated in the simplest way by considering β to be the independent variable. Then D can be solved from (4.14c), whereafter $\bar{\epsilon}/\epsilon_0$, $F/\bar{\epsilon}$, P/ϵ_0 and ϵ/ϵ_0 can be established with Eqs. (4.14a, b) (4.15) and (4.10).

Obviously, the P–ϵ-relation so established depends on $\alpha T(y)/\epsilon_0$. Likewise, as in the buckling problem, temperature variations have a smaller effect when they occur close to the edges. When studying the thermal effects on post-buckling behaviour by means of numerical examples such temperature distribution should be assumed which yields a pronounced thermal effect; therefore the temperature is assumed to vary across a large percentage of the plate width. In this respect the distribution,

$$T=T_1\sin\pi\, y/b \qquad (4.16)$$

is appropriate. Usually in those stages of flight in which the temperature difference between the centre and the edges of the wing panel are greatest, the temperature gradient near the edges is larger than according to a sine-law, yielding more uniformity of thermal strain in the centre region. Therefore the distribution (4.16) is on the conservative side.

In order to get some idea of the thermal to buckling strain ratios $(\alpha T_1)/\epsilon_0$ to be expected in wing structures the following considerations are given.

We consider an aluminium-alloy structure, where $\alpha=13{\cdot}3\times10^{-6}$ °C^{-1}, $E=6\times10^3$ kg/mm² at elevated temperature, the ultimate compressive strength be $\sigma_u=40$ kg/mm². The temperature T_1 is assumed to be 150°C, therefore $\alpha T_1=2{\cdot}10^{-3}$. Since the skin of supersonic wings should not buckle in normal (unaccelerated) flight the buckling stress σ_0 at constant temperature will not be less than 10 kg/mm². Hence $(\alpha T_1)/\epsilon_0<1{\cdot}2$.

Therefore when evaluating the range $\beta<1$ we should consider $(\alpha T_1)/\epsilon\approx 1{\cdot}0$. We will however pay some attention as well to the ratios 2 and 3.

Table I gives the relation between P and ϵ following from (4.14), (4.15) and (4.16). We will call this solution the exact solution and we will compare it with an approximate solution $P_1(\epsilon)$. This approximation is obtained under the assumption that the equality $P_1-P_b=\bar{P}-\bar{P}_b$ for $\epsilon-\epsilon_b=\bar{\epsilon}-\bar{\epsilon}_b$ holds not only in the range where $\beta=1$ but as well in the range where $\beta<1$.

The corresponding values $P_1/Etb\epsilon_0$ and ϵ/ϵ_0 are obtained from $\overline{P}/Etb\epsilon_0$ and $\bar{\epsilon}/\epsilon_0$ for $T=0$ by moving the $\overline{P}/Etb\epsilon_0$ axis over the distance $\epsilon_T/\epsilon_0=8/3\pi(\alpha T_1)/\epsilon_0$ and the $\bar{\epsilon}/\epsilon_0$ axis over the distance $\epsilon_T/\epsilon_0-(\alpha T/\epsilon_0)_{\text{av}}=2/3\pi\,(\alpha T_1)/\epsilon_0$.

TABLE 1

P–ε-relation for the advanced post-buckling range

$\frac{\alpha T_1}{\epsilon_0}$	0			1·0				
β	D	$\frac{\bar{\epsilon}}{\epsilon_0}$	$\frac{\overline{P}}{Etb\epsilon_0}$	D	$\frac{\epsilon}{\epsilon_0}$	$\frac{P}{Etb\epsilon_0}$	$\frac{P_1}{Etb\epsilon_0}$	$\frac{P_1-P}{P}\times 100$
1	1·773	3·985	2·25	1·865	3·75	2·28	2·26	−1
0·875	2·194	5·82	2·88	2·285	5·73	2·93	2·90	−1
0·75	2·816	8·87	3·77	2·909	8·98	3·88	3·82	−1·5
0·625	3·81	14·53	5·21	3·92	14·82	5·30	5·20	−1·9
0·50	5·615	26·51	7·60	5·70	26·96	7·74	7·69	−0·7
0·375	9·35	57·5	12·72	9·43	58·44	12·83	12·76	−0·6
0·25	19·71	177	26·1	19·78	178·6	26·4	26·2	−1·0

$\frac{\alpha T_1}{\epsilon_0}$	β	D	$\frac{\epsilon}{\epsilon_0}$	$\frac{P}{Etb\epsilon_0}$	$\frac{P_1}{Etb\epsilon_0}$	$\frac{P_1-P}{P}\times 100$
	1	1·934	3·39	2·25	2·25	0
2·0	0·875	2·357	5·51	2·93	2·88	−2
	0·75	2·985	8·95	3·92	3·82	−2·8
3·0	0·75	3·061	8·96	4·00	3·81	−4·8

Table 1 gives the comparison of this approximation with the exact solution. It appears that the error is always on the conservative side. It passes through a maximum which increases with increasing $(\alpha T_1)/\epsilon_0$. For $(\alpha T_1)/\epsilon_0=1\cdot 0$ which is for aluminium structures the severest case to be considered the maximum error does not exceed 2%. However, this maximum is reached at $\epsilon/\epsilon_0>10$, therefore when the edge stress is more than 10 times buckling stress. This edge stress is not realistic, since we had assumed that the ratio of ultimate strength and buckling stress would not exceed 4. With $\epsilon/\epsilon_0=4$ the error is about 1%.

If we cancel the condition that the skin should not buckle in continuous flight and if we allow for buckling stresses down to 4 kg/mm², keeping the ultimate strength at 40 kg/mm² and the temperature T_1 at 150°C we obtain the errors given in Table 2. These are estimated figures derived from Table 1. Only when we get down to the unrealistic figure $\sigma_0 = 4$ kg/mm² the approximation becomes too rude for practical application. For practical conditions however the error will not exceed 1% and

TABLE 2

Error of approximate solution P_1

σ_0	$\frac{\epsilon}{\epsilon_0}$	$\frac{\alpha T_1}{\epsilon_0}$	$\frac{P_1-P}{P}\times 100$
12	3·33	1·0	0
10	4·0	1·2	−0·8
8	5·0	1·5	−1·1
6	6·67	2·0	−2·3
5	8	2·4	−3·5
4	10	3·0	−5

it is still less for more realistic temperature distributions than have been assumed in this analysis. This means that the approximation P_1 satisfies engineering needs; not only in the first post-buckling stage but also in the more advanced range the P–ϵ curves beyond the onset of buckling are identical in the thermal and the isothermal case.

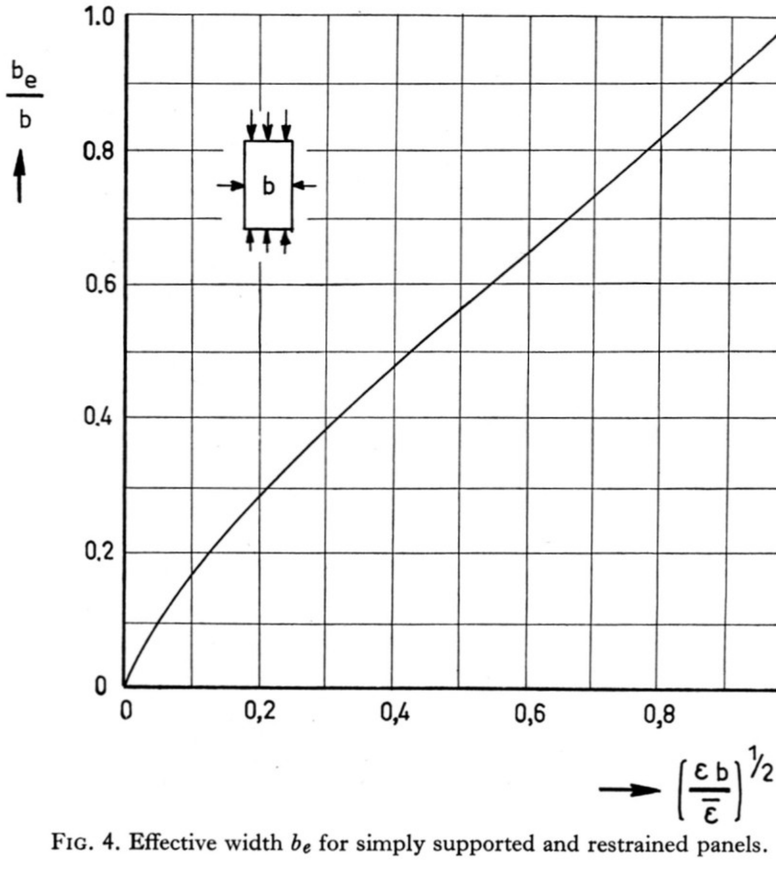

FIG. 4. Effective width b_e for simply supported and restrained panels.

4.4. *Post-buckling Behaviour of Plates with Restrained Edges*

The analysis considered so far concerns the simply supported plate. In the case of restrained edges the deflections in the edge regions are smaller. This reduces the effect of temperature gradients in the edge regions on the phenomenon of buckling and consequently likewise on post-buckling behaviour. Hence the procedure for establishing the post-buckling characteristics of simply supported panels under thermal loads is with still better accuracy applicable to the case of restrained panels. The relation between the P–ϵ-curves for the thermal and the isothermal case is again given by (4.13).

With rigidly restrained edges the functions (a) or (b) for W can be used. Function (b) is more accurate, since the errors in buckling stress corresponding to the approximations (a) and (b) are 4·5 and 0·7% respectively[4].

$$\text{(a)}\ W = \sin^2 \pi\, y/b, \text{ yielding} \int_0^b W^2(y)\, \mathrm{d}\, y/b = \frac{3}{8}$$

$$\text{(b)} \qquad 0 < y < \frac{1}{6}\mathrm{b} : W = \frac{1}{3}(1 - \cos 3\pi\, y/b) \tag{4.17}$$

$$\frac{1}{6}b < y < \tfrac{1}{2}b : W = \frac{1}{3} + \frac{2}{3}\sin\left(\frac{3}{2}\pi\, y/b - \pi/4\right)$$

$$\int_0^b W^2(y)\, \mathrm{d}\frac{y}{b} = 0{\cdot}41925$$

These data allow the distances between the axes of $\bar{P}$ and P and between the axes of $\bar{\epsilon}$ and ϵ to be established for any given temperature distribution.

TABLE 3

Simply supported plate, wave pattern and load (see Figs. 4 and 5)

β	$\frac{L}{b}$	$\frac{f}{t}$	$\left(\frac{\epsilon_0}{\bar{\epsilon}}\right)^{\frac{1}{2}}$	$\frac{\bar{P}}{Etb\epsilon_0}$	$\frac{\bar{P}}{Etb\bar{\epsilon}} = \frac{b_e}{b}$	$\frac{\bar{P}}{Etb\bar{\epsilon}}$ Eq. (4.18)
1	1	0	1	1	1	1
1	0·953	0·529	0·910	1·103	0·913	0·913
1	0·894	0·866	0·789	1·286	0·801	0·802
1	0·816	1·291	0·629	1·68	0·663	0·665
1	0·756	1·658	0·510	2·19	0·571	0·568
1	0·751	1·690	0·501	2·25	0·565	0·561
0·875	0·675	1·868	0·4144	2·88	0·495	0·489
0·75	0·596	2·06	0·3358	3·77	0·425	0·420
0·625	0·512	2·29	0·2623	5·19	0·357	0·354
0·50	0·422	2·56	0·1942	7·60	0·287	0·285
0·375	0·327	2·94	0·1319	12·72	0·221	0·219
0·25	0·225	3·58	0·0752	26·1	0·147	0·142

The $\overline{P}$–$\bar{\epsilon}$-curve (Fig. 4) is represented very well, both for simply supported and restrained edges, by the formula given in ref. 4 (see Table 3)

$$\frac{\overline{P}}{Etb} = \bar{\epsilon}\left\{1{,}20\left(\frac{\bar{\epsilon}_b}{\bar{\epsilon}}\right)^{2/5} - 0{,}65\left(\frac{\bar{\epsilon}_b}{\bar{\epsilon}}\right)^{4/5} + 0{,}45\left(\frac{\bar{\epsilon}_b}{\bar{\epsilon}}\right)^{6/5}\right\} = \bar{\epsilon}\,\frac{b_e}{b}. \qquad (4.18)$$

The quantity b_e is the "effective width".

4.5 *Wave Depth, Wavelength, and Equivalent Stress*

As a consequence of the equality of behaviour beyond the onset of buckling under thermal and isothermal conditions, data on waviness and stresses caused by buckling established for the isothermal case can be applied to thermal conditions. Such information is available in ref. 4, where f/t and L/b have been given for simply supported and rigidly restrained panels. These data as well as β are given in Fig. 5 and Table 3.

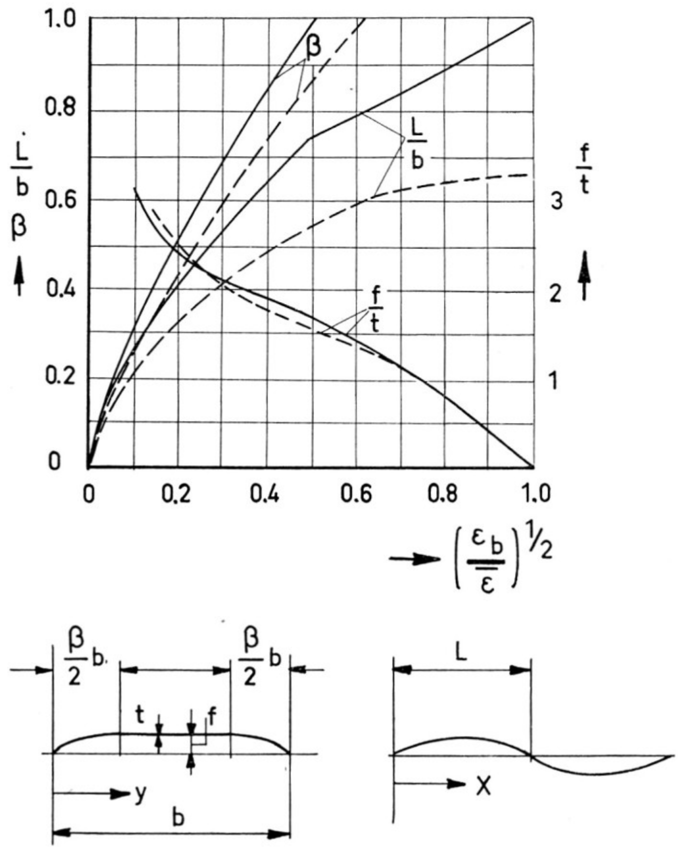

FIG. 5. Wave depth f, half wavelength L, and width $\frac{1}{2}\beta b$ of double curved edge for simply supported (full lines) and restrained panels (dotted lines).

The stresses imposed by buckling are not evaluated in ref. 4, but they can be established on the basis of the displacement functions u, v, w. This will be done here for the case of simply supported panels.

If we put

$$p = \pi\, x/L,\; q = \pi \frac{y}{\beta b}$$

the stress components are

in the edge region $0<y<\frac{1}{2}\beta b$:

$$\sigma_x/E\bar{\epsilon} = -(\epsilon + \alpha T)/\bar{\epsilon} + X_1 \sin^2 q - X_2 \cos 2p \pm X_3 \sin p \sin q$$

$$\sigma_y/E\bar{\epsilon} = -Y_1 \cos 2p \pm Y_2 \sin p \sin q$$

$$\tau/E\bar{\epsilon} = \mp Z \cos p \cos q$$

and in the centre region $\frac{1}{2}\beta b<y<(1-\frac{1}{2}\beta)b$:

$$\sigma_x/E\bar{\epsilon} = -(\epsilon + \alpha T)/\bar{\epsilon} + X_1 \pm X_4 \sin p$$

$$\sigma_y/E\bar{\epsilon} = \pm X_3 \sin p \qquad (4.19)$$

$$\tau/E\bar{\epsilon} = 0.$$

The sign $\pm$ or $\mp$ applies to the bending or torsional part of the stresses; the upper sign refers to the top surface, the lower sign to the lower surface, the hollow side, of the panel.

The coefficients X, Y, Z are functions of the strain ratio $\bar{\epsilon}/\epsilon_0$ and have been given in Table 4 and Fig. 6. For $\bar{\epsilon}/\epsilon_0$ very close to unity X_3, Y_2, Z

TABLE 4

Simply supported plate, stress coefficients (see Figs. 6 and 7)

$\left(\frac{\epsilon_0}{\bar{\epsilon}}\right)^{1/2}$	β	X_1	X_2	X_3	X_4	Y_1	Y_2	Y_3	Z
1	1	1	0	0	—	0	0	—	0
0·910	1	0·174	0·026	0·920	—	0·087	0·874	—	0·483
0·789	1	0·399	0·053	1·254	—	0·176	1·113	—	0·635
0·629	1	0·675	0·074	1·380	—	0·247	1·111	—	0·658
0·510	1	0·855	0·081	1·328	—	0·269	0·990	—	0·602
0·501	1	0·870	0·081	1·324	(1·134)	0·270	0·977	(0·339)	0·596
0·4144	0·875	0·898	0·088	1·246	1·058	0·295	0·950	0·318	0·570
0·3358	0·75	0·921	0·096	1·173	0·983	0·320	0·918	0·295	0·547
0·2623	0·625	0·935	0·104	1·082	0·901	0·345	0·876	0·270	0·516
0·1942	0·50	0·950	0·112	0·991	0·817	0·372	0·826	0·245	0·482
0·1319	0·375	0·960	0·121	0·901	0·720	0·402	0·763	0·216	0·438
0·0752	0·25	0·973	0·130	0·743	0·598	0·434	0·665	0·179	0·378

and f/t change rapidly with $\bar{\epsilon}/\epsilon_0$. Fig. 7 gives for this range these quantities vs. $x=[1-(\epsilon_0/\bar{\epsilon})^{\frac{1}{2}}]^{\frac{1}{2}}$, since for $x=0$ the tangents to the curves are given by

$$X_3 = Y_2 = \left(\frac{6}{1-\frac{3}{4}\nu^2}\right)^{\frac{1}{2}}(1+\nu)\,x = 3{\cdot}297\,x$$

$$Z = \frac{1-\nu}{1+\nu}X_3 = 1{\cdot}775\,x \qquad (4.20)$$

$$\frac{f}{t} = \left(\frac{8}{3\,(1-\frac{3}{4}\nu^2)}\right)^{\frac{1}{2}}x = 1{\cdot}691x.$$

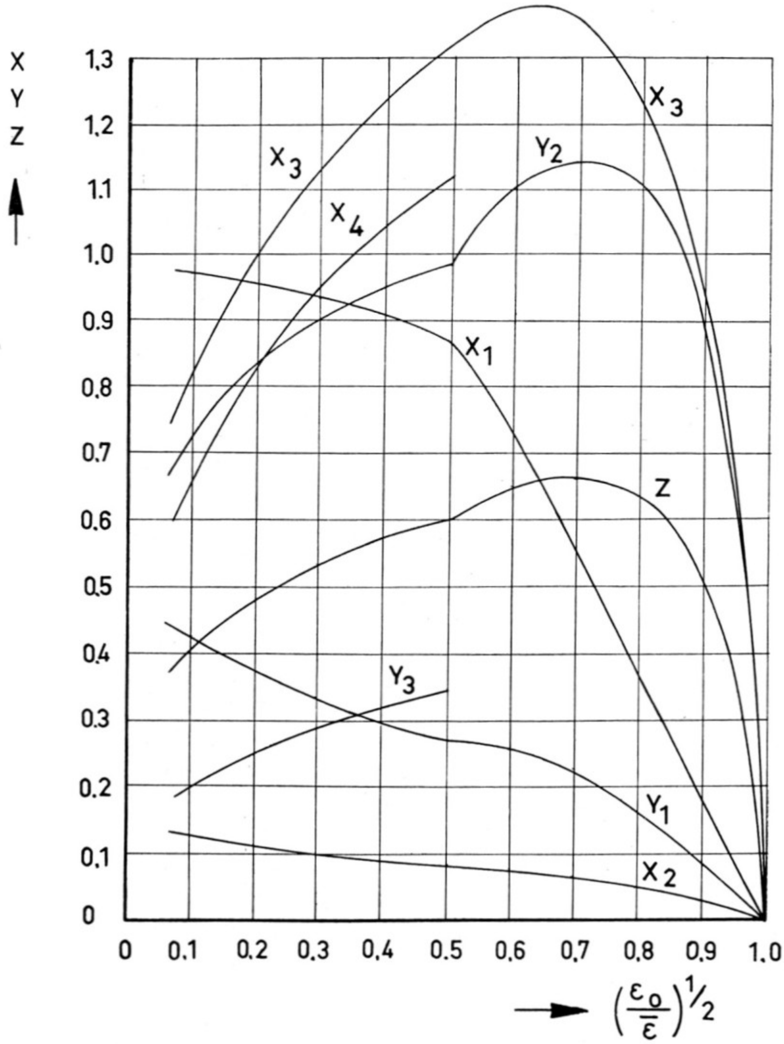

FIG. 6. Stress coefficients for simply supported panels.

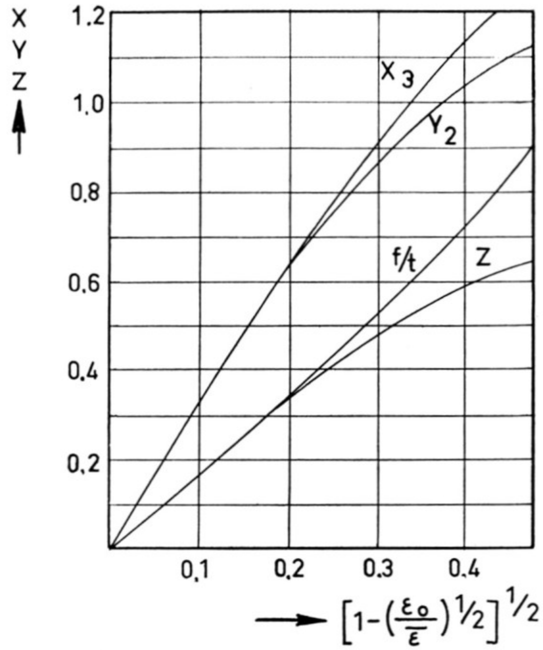

FIG. 7. Stress coefficients for simply supported panels in the primary post-buckling range.

Since $E\bar{\epsilon}$ is the stress in the plate when buckling would be prevented the ratios $\sigma_x/E\bar{\epsilon}$, etc., are the ratios between the stress components in the buckled plate to the stress in the unbuckled plate at the same longitudinal strain $\bar{\epsilon}$.

If we derive the equivalent stress σ_a from the Huber–von Mises–Hencky yield criterion

$$\sigma_a = [(\sigma_x - \sigma_y)^2 + \sigma_x\sigma_y + 3\tau^2]^{\frac{1}{2}} \tag{4.21}$$

the allowable compressive strain ϵ follows from the condition that the maximum of σ_a defined by (4.21) is equal to the yield limit.

σ_x depends on $\epsilon+\alpha T=\bar{\epsilon}-\epsilon_T(\alpha T)+\alpha T$. Then σ_a and its maximum depend as well on the temperature distribution. Therefore (4.21) cannot be evaluated once for all. For any given temperature distribution however σ_a can be established using the formulae (4.19) and (4.21) and the data given in Figs. 6 and 7 or Table 4.

The situation in the isothermal case gives a useful guide for a conjecture on the points where the equivalent stress is maximal.

In the case $\alpha T=0$ the points A, B, C indicated in Fig. 8 have large stresses. $A(p=q=0)$ is the intersection of the nodal line and the edge; apart from the stress σ_x the torsional stress is large. B and C ($p=\pi/2$; $q=\pi/2$) are situated on the wave crest at the boundary of the double curved edge region and the developable centre region, at the hollow side

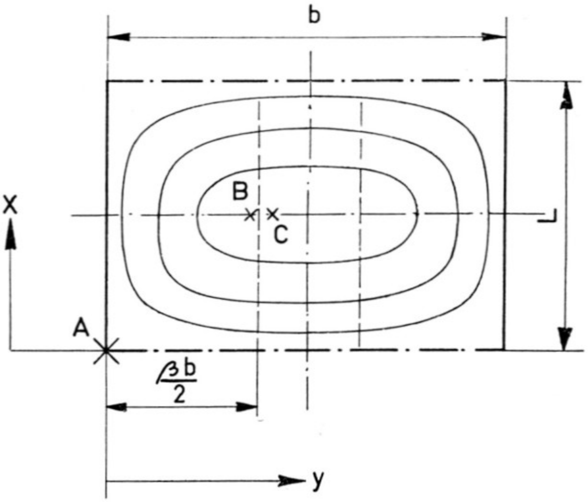

FIG. 8. Places of maximal equivalent stress.

of the bulge. Due to the discontinuity in the assumed deformation at this point the stresses in B and C, which are in fact equal, will appear to be slightly different. The actual stress can be assumed to be the average of the stresses in B and C at the hollow (lower) side of the plate. This applies of course only to deformations where $\beta<1$. For $\bar{\epsilon}/\epsilon_0<4$ the point C does not exist. For $\bar{\epsilon}/\epsilon_0>2$ the maximal equivalent stress occurs in A; for $\bar{\epsilon}/\epsilon_0<2$ B,C is the point with maximal stress.

The addition of temperature gradients changes the stresses; the stress decreases in A and increases in B,C. Therefore B,C will be critical in a larger range of $\bar{\epsilon}/\epsilon_0$. As far as the vicinity of A is critical it should be taken into account that the slope of αT close to the edge is large; this means that the maximum of σ_a may be located on the nodal line at some distance from the edge.

5. POST-BUCKLING BEHAVIOUR UNDER SIMULTANEOUS COMPRESSION AND SHEAR

In comparison with the isothermal case, the situation in the thermal case differs in the rapid change of the longitudinal stress in the edge region. The "correction method" for establishing buckling loads has been based on the idea that the stress σ_x in the region close to the edge does not affect the stability significantly, due to the smallness of $(\partial w/\partial x)^2$ in this region. This suggested that in general the wave pattern would undergo a negligible change through the addition of thermal stresses, as appeared to be true in the case of longitudinally compressed panels.

This idea has its importance as well with respect to the post-buckling problem, as has been confirmed by the investigation on the longitudinally compressed plate in Section 4.

If then at finite deflections the wave patterns in the thermal and in the isothermal case are equal, the stress increments beyond buckling following from incremental edge displacements beyond those at the onset of buckling are unaffected by thermal conditions. The incremental edge loads are the resultants of the incremental stresses. Therefore the load increment beyond buckling load is independent of the thermal state of the plate and data on post-buckling behaviour for isothermal conditions are valid as well for thermal conditions.

The buckling load in the thermal case can be established by means of Eq. (2.10). The load increment beyond buckling load can be taken from the solution for the isothermal case applying the mechanical properties of the material at the ambient temperature level.

Information on post-buckling behaviour of simply supported panels under simultaneous longitudinal and lateral compression and shear is available in refs. 8 and 9.

6. CONCLUSIONS

Non-uniformity of longitudinal stress due to temperature gradients in the region close to the edge has a negligible effect on the wave pattern. Hence the buckling mode of uniformly stressed plates is a very good approximation for the buckling mode under thermal stresses. This yields a straightforward method for establishing the buckling load in the thermal case if the solution for uniform stresses is available (Eq. 2.10).

Post-buckling behaviour of thermally loaded and longitudinally compressed panels proves to be negligibly different from post-buckling behaviour in the isothermal case for the practical range of ratios between thermal strain (αT) and buckling strain. This suggests that for post-buckling behaviour also, the wave pattern is negligibly affected by thermal strain. This approach yields that the relation between load increments and edge displacement increments beyond the onset of buckling is identical for thermal and isothermal conditions.

Concerning thermally loaded longitudinally compressed panels, data are given on the relation between compressive load and edge strain, on wave depth, wavelength and stresses as a function of edge strain.

For the case of thermally loaded panels under simultaneous longitudinal and lateral compression and shear use can be made of available information on post-buckling behaviour in the isothermal case.

7. REFERENCES

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DISCUSSION

N. J. HOFF*: The writer agrees with Professor van der Neut's statement in his excellent paper that a remarkable feature of the thermal buckling problem is that the deflected shape differs only slightly from the sinusoidal form even though the stress distribution is markedly non-uniform. This fact can be brought out more completely if the differential equation, Eq. 2.1, is solved rigorously by the inverse method.

Let it be assumed that σ_y and τ are identically zero. The solution can be written as

$$w = Y \sin(\pi x/L)$$

where Y is a function of y only. Substitution yields the ordinary differential equation

$$B[Y^{IV} - 2(\pi/L)^2 Y'' + (\pi/L)^4 Y] = -\sigma_x t(\pi/L)^2 Y$$

If the deflected shape is taken as

$$Y = A[\sin(\pi y/b) - p \sin(3\pi y/b)]$$

the ordinary differential equation is identically satisfied if

$$\sigma_x = -\frac{\pi^2 E}{12(1-\nu^2)} \left(\frac{t}{L}\right)^2 F(y)$$

where

$$F(y) = \frac{[(L/b)^2 + 1]^2 \sin(\pi y/b) - p[(3L/b)^2 + 1]^2 \sin(3\pi y/b)}{\sin(\pi y/b) - p \sin(3\pi y/b)}$$

When $p = 0$, this expression reduces to

$$\sigma_x = -\frac{\pi^2 E}{3(1-\nu^2)} \left(\frac{t}{L}\right)^2$$

if the plate is square, that is if $L/b = 1$. Other assumptions for p yield the curves given in Fig. 9.

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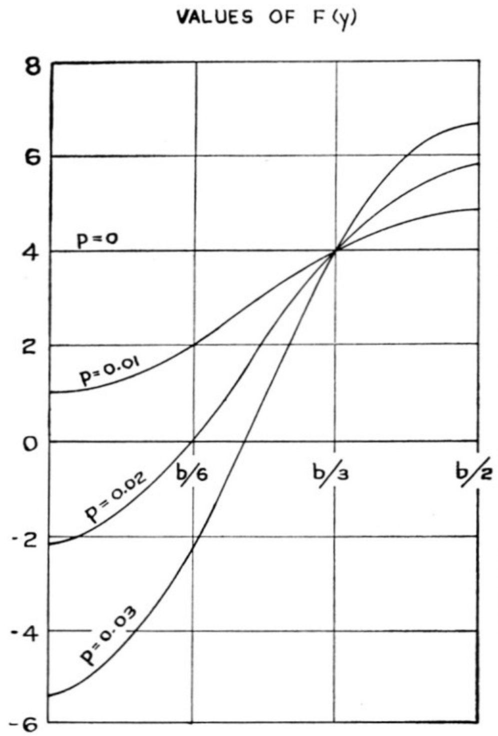

FIG. 9. Critical stress distributions.

It can be seen that remarkably small deviations from the sinusoidal shape lead to very large variations in the critical stress distribution. This explains the success of the approximate analyses carried out by the author.